Week 1 (Jan 11)

MATH 2020 B

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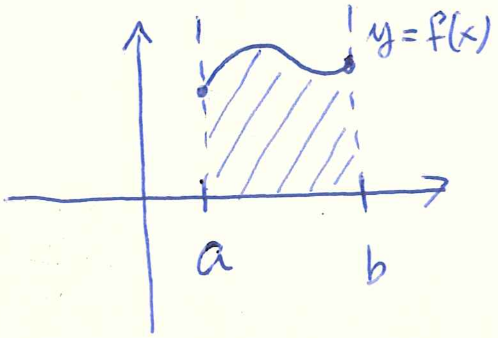
"Single Integral"

Let f be a function on interval $[a,b]$. Want to define

$$\int_a^b f(x)\,dx.$$

Motivation: when $f \geq 0$, $\int_a^b f(x)\,dx$ is the area of the region bdd by the graph of f over the interval $[a,b]$.



area of the shaded region D

$$= \int_a^b f(x)\,dx.$$

- A partition P is a collection of points

$$x_0, x_1, \ldots, x_n,$$

satisfying $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.

- A tag is a choice of $\{z_1, z_2, \ldots, z_n\}$, $z_j \in [x_{j-1}, x_j]$, $j = 1, 2, \ldots n$.

- The Riemann sum of f with respect to a partition P with tags :

$$R(f,P) = \sum_{j=1}^{n} f(z_j)\,\Delta x_j, \quad \Delta x_j = x_j - x_{j-1}.$$

When $f \geq 0$, $R(f,P)$ is an approximate area of D.

- The norm of P, $\|P\|$, which measures how refined the partition is, is $\max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$.

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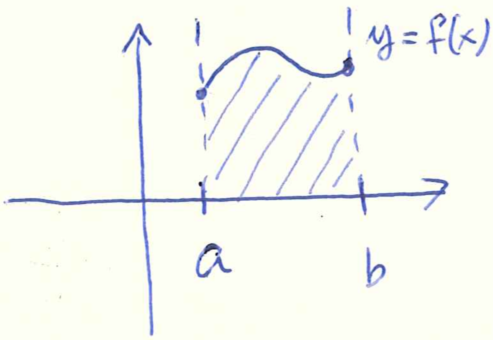

area of the shaped region D $= \int_a^b f(x)\,dx$.

- A partition P is a collection of points
  $$x_0, x_1, \dots, x_n,$$
  satisfying $x_0 = a < x_1 < x_2 < \dots < x_n = b$.
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When $f \geq 0$, $R(f, P)$ is an approximate area of D.

- The norm of P, $\|P\|$, which measures how refined the partition is, is $\max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$.

When $R(f,P)$ approaches a certain number as $\|P\| \to 0$ regardless the choice of tags, we call this number the (Riemann) Integral of f over $[a,b]$, denoted by

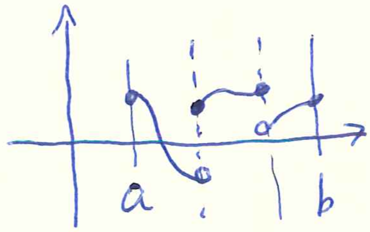
$$\int_a^b f, \quad \int_a^b f(x)\,dx, \quad \text{or} \quad \int_{[a,b]} f, \quad \text{etc.}$$

When $f \geq 0$, it is the area of D.

A function whose Riemann sum admits this property is called an integrable function.

Facts concerning integrable functions:

- All continuous functions are integrable,
- Further, all piecewise continuous functions are also integrable,
- Unbounded functions are not integrable.
- The following bounded function is not integrable.



the graph of a piecewise continuous function.

Let

$$\varphi(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$$

φ is not integrable on any $[a,b]$.

(Jan 13)

Double Integral

The theory of double integral is similar to that of the single integral.

Let $f(x,y)$ be a function on rectangle $[a,b]\times[c,d]$.

- A partition P is a collection of points $\{x_0, x_1, \cdots, x_n, y_0, y_1, \cdots, y_m\}$ satisfying

$$a = x_0 < x_1 < \cdots < x_n = b,$$

$$c = y_0 < y_1 < \cdots < y_m = d.$$

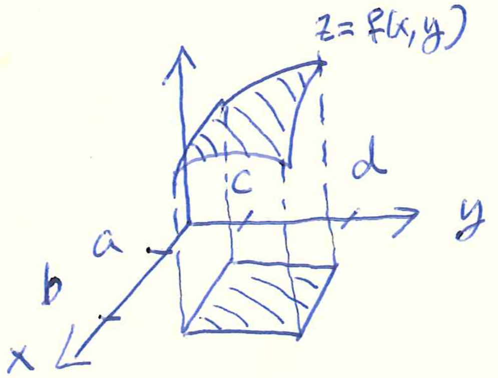
P divides $R=[a,b]\times[c,d]$ into subrectangles

$$R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j], \quad i=1,2,\cdots,n, \quad j=1,2,\cdots,m.$$

- A tag is a choice of P_{ij}, $P_{ij}\in R_{ij}$.
- The Riemann sum of f with respect to P and tags

$$R(f,P) = \sum_{i,j=n,m} f(P_{ij})\,\Delta x_i \Delta y_j$$

when $f\geq 0$, $R(f,P)$ is the approximate volume of Ω, ie, the solid region bounded by the graph of f over R.



the region Ω

- $\|P\| = \max\{\Delta x_1, \cdots \Delta x_n, \Delta y_1, \cdots, \Delta y_m\}$

When $R(f, P)$ approaches a certain number as $\|P\| \to 0$ regardless the choice of tags, we call this number the integral of f over R, denoted by

$$\iint_R f, \quad \iint_R f(x,y)\, dA, \quad \iint_R f(x,y)\, dA(x,y), \text{ etc.}$$

When $f \geq 0$, it is the volume of Ω.

A function admitting an integral is called an integrable function.

Facts concerning integrable functions are the same as in the single integral case.

The evaluation of double integral is based on the following theorem.

<u>Fubini's Theorem</u> Let f be a continuous function on $R = [a,b] \times [c,d]$. Then

$$\iint_R f = \int_c^d \left(\int_a^b f(x,y)\, dy \right) dx$$

$$= \int_a^b \left(\int_c^d f(x,y)\, dx \right) dy.$$

e.g. Evaluate $\displaystyle\iint_R x y^2\, dA$, $R = [0,2] \times [0,1]$.

By Fubini's theorem,

$$\iint_R xy^2\,dA = \int_0^2 \left(\int_0^1 xy^2\,dy \right) dx$$

$$= \int_0^2 \frac{xy^3}{3} \Big|_{y=0}^{y=1} dx$$

$$= \int_0^2 \frac{x}{3}\,dx$$

$$= \frac{x^2}{6} \Big|_0^2$$

$$= \frac{2}{3}.$$

Alternatively,

$$\iint_R xy^2\,dA = \int_0^1 \left(\int_0^2 xy^2\,dx \right) dy$$

$$= \int_0^1 \frac{x^2}{2} y^2 \Big|_{x=0}^{x=2} dy$$

$$= \int_0^1 2y^2\,dy$$

$$= \frac{2}{3} y^3 \Big|_0^1$$

$$= \frac{2}{3}.$$

e.g. Evaluate $\iint_R x \sin xy\,dA$, $R = [0,1] \times [\pi/2, \pi]$.

$$\iint_R x \sin xy\,dA = \int_{\pi/2}^{\pi} \left(\int_0^1 x \sin xy\,dx \right) dy.$$

$$\int_0^1 x \sin xy \, dx = \int_0^1 x \left(-\frac{\cos xy}{y}\right)' dx$$

$$= x\left(-\frac{\cos xy}{y}\right)\Big|_0^1 - \int_0^1 \frac{-\cos xy}{y} dx$$

$$= -\frac{\cos y}{y} + \int_0^1 \frac{\cos xy}{y} dx$$

$$= -\frac{\cos y}{y} + \frac{\sin y}{y^2}.$$

So, $\iint_R x \sin xy \, dA = \int_{\pi/2}^{\pi} \left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right) dy$, get stuck!

So, consider the other way,

$$\iint_R x \sin xy \, dA = \int_0^1 \left(\int_{\pi/2}^{\pi} x \sin xy \, dy\right) dx.$$

$$\int_{\pi/2}^{\pi} x \sin xy \, dy = -\cos xy \Big|_{y=\frac{\pi}{2}}^{y=\pi}$$

$$= -\cos \pi x + \cos \frac{\pi}{2} x.$$

$$\iint_R x \sin xy \, dA = \int_0^1 \left(-\cos \pi x + \cos \frac{\pi}{2} x\right) dx$$

$$= \left(-\frac{\sin \pi x}{\pi} + \frac{2}{\pi} \sin \frac{\pi}{2} x\right)\Big|_0^1$$

$$= \frac{2}{\pi}.$$

e.g. Find the volume of the region bounded above by $z = 10 + x^2 + 3y^2$ over the rectangle $0 \le x \le 1, 0 \le y \le 2$.

$$\text{Vol} = \iint_{[0,1]\times[0,2]} (10 + x^2 + 3y^2) \, dA(x,y)$$

$$= \int_0^1 \int_0^2 (10 + x^2 + 3y^2) \, dy \, dx$$

$$= \int_0^1 (10y + x^2y + y^3)\Big|_0^2 dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx$$

$$= \left(28x + \frac{2x^3}{3}\right)\Big|_0^1$$

$$= 86/3.$$

SOME theory.

"Proof of Fubini's thm" Choose tag $P_{ij} = (x_i, y_j)$.

$$\iint_R f \approx R(f,P) = \sum_{i,j}^{n,m} f(P_{ij}) \Delta x_i \Delta y_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta y_j \Delta x_i$$

$$\approx \sum_{i=1}^{n} \int_c^d f(x_i, y)\, dy\, \Delta x_i$$

$$\approx \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

Basic Properties

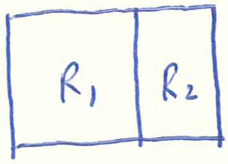
1. linearity

$$\iint_R (af + bg)\, dA = a \iint_R f\, dA + b \iint_R g\, dA$$

2. positivity: $f \geq 0$ implies

$$\iint_R f\, dA \geq 0.$$

③ additivity : R divides onto R_1, R_2



$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA .$$

"Pf of ①"

$$\begin{aligned}
\iint_R (af+bg)\,dA &\approx R(af+bg) \\
&= \sum_{i,j} \left(af(P_{ij}) + bg(P_{ij})\right) \Delta x_i \Delta y_j \\
&= a \sum_{i,j} f(P_{ij}) \Delta x_i \Delta y_j + b \sum_{i,j} g(P_{ij}) \Delta x_i \Delta y_j \\
&\approx a \iint_R f\,dA + b \iint_R g\,dA .
\end{aligned}$$